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GROUP REDUCTION OF THE LAMÉ EQUATIONS*

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Group reduction is realized with respect to a certain infinite Lie group of transformations of the Lamé equations of the static theory of elasticity, which enables us to represent them as a combination of two systems of first-order differential equations: automorphic and resolving. The general solution found for the automorphic system is a multidimensional analogue of the Kolosov-Muskhelishvili formula. In the three-dimensional case the resolving system turns out to be conformally-invariant, while the Lamé equations themselves allow only a similarity group of three-dimensional Euclidean space. Because of the conformal invariance, transformations analogous to the Kelvin transformation exist for the resolving system. The general form of such transformations is presented. The structure of the resolving system enables us to introduce complex variables in a natural way in the three-dimensional case, which is convenient for constructing classes of exact solutions. Investigation of the group properties of the Lamé equations of classical elasticity theory started on the initiative of Ovsyannikov in /1/ in which the three-dimensional dynamic equations were examined. The static Lamé equations are studied below in an arbitrary space R^n by group-analysis methods.

1. Group reduction.

The equilibrium state of a homogeneous elastic medium is described, when there are no mass forces, by the system of Lamé equations

$$(\lambda+\mu)\nabla \operatorname{div}\mathbf{u}+\mu\Delta\mathbf{u}=0 \tag{1.1}$$
$$(\mathbf{u}=(u_1, u_2, \ldots, u_n),\ \mathbf{x}=(x_1, x_2, \ldots, x_n))$$

The displacement vector $\mathbf{u}$ is a function of the point $\mathbf{x}$ while $\lambda>0,\ \mu>0$ are Lamé constants.

The broadest Lie transformation group of the space $R^{2n}(\mathbf{x},\mathbf{u})$ allowed by this system is obviously infinite-dimensional since the transformation

$$\mathbf{u}\to\mathbf{u}+\mathbf{u}_0(\mathbf{x}) \tag{1.2}$$

($\mathbf{u}_0(\mathbf{x})$ is an arbitrary solution of (1.1)) conserves the system. These transformations form an infinite normal divisor over which the factor-group is finite-dimensional. The corresponding Lie algebra of the operators is found by a standard method /2/ and has the basis

$$\partial_\mathbf{x},\ \mathbf{x}\cdot\partial_\mathbf{x},\ \mathbf{u}\cdot\partial_\mathbf{u},\ A\langle\mathbf{x}\rangle\cdot\partial_\mathbf{x}+A\langle\mathbf{u}\rangle\cdot\partial_\mathbf{u}$$

Here A is an arbitrary antisymmetric transformation of the space R^n while the angular brackets denote the transform under a linear mapping.

Corresponding to each vector operator there are n scalars, for instance $\partial_x=(\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n)$.

We know /2/ that if a system of differential equations admits of a certain group, then the action of this group on the set of solutions of the system results in its partition into a class of equivalent solutions. The structure originating in the set of solutions is described as a result by two systems of differential equations: automorphic and resolving. The automorphic system characterises separate classes of equivalent solutions (the group acts transitively on its solutions), while the resolving system is the set of all such classes (the group acts identically on its solutions). Representation of the system in the form of its equivalent union of the automorphic and resolving systems is called group reduction of the system relative to this group.

Let us carry out group reduction of (1.1) with respect to the infinite subgroup, generated by the operators $\nabla h(x)\cdot\partial_u$ contained in the normal divisor (1.2), where $h(x)$ is an arbitrary harmonic function. We will first obtain the automorphic system. The invariants

$$\mathbf{I}_1=\mathbf{x},\quad I_2=\operatorname{div}\mathbf{u},\quad I_3=\partial_\mathbf{x}\mathbf{u}-(\partial_\mathbf{x}\mathbf{u})^*$$

can be selected as the basis of the differential invariants of the subgroup under consideration.

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Denoting the invariants I_2 and I_3 as functions of the invariant $\mathbf{I}_1$, we have the automorphic system

$$(2+\lambda/\mu)\operatorname{div}\mathbf{u}=\theta(\mathbf{x}),\quad \partial_{\mathbf{x}}\mathbf{u}-(\partial_{\mathbf{x}}\mathbf{u})^*=\omega(\mathbf{x}) \tag{1.3}$$

(the constant factor $(2+\lambda/\mu)$ in front of $\operatorname{div}\mathbf{u}$ is introduced to simplify the subsequent formulas).

The resolving system has the form

$$\begin{aligned}&\nabla\theta+\operatorname{div}\omega=0\\ &\mathbf{a}\cdot\partial\omega\langle\mathbf{b},\mathbf{c}\rangle+\mathbf{b}\cdot\partial\omega\langle\mathbf{c},\mathbf{a}\rangle+\mathbf{c}\cdot\partial\omega\langle\mathbf{a},\mathbf{b}\rangle=0\end{aligned} \tag{1.4}$$

where $\mathbf{a},\mathbf{b},\mathbf{c}$ are arbitrary (trial) vectors from R^n.

Let us transform the Lamé equation by using the concept of tensor divergence /3/. By virtue of the identity

$$\Delta u=\nabla\operatorname{div}u+\operatorname{div}(\partial_{\mathbf{x}}u-(\partial_{\mathbf{x}}u)^*)$$

the condition of compatibility of system (1.3) with system (1.1) will be the first equation of system (1.4). Its second equation is the condition of compatibility of system (1.3).

The desired group reduction of the Lamé Eqs.(1.1) is the union of the automorphic system (1.3) and the resolving system (1.4).

For $n=2$, (1.4) is a Cauchy-Riemann system, which indeed permits successful application of methods of complex variable function theory in plane static elasticity theory problems.

2. Solution of the automorphic system. Any solution of the automorphic system (1.3) is obtained from one of its fixed solutions by transformation of the subgroup with respect to which the group reduction is performed. Consequently, the general solution of this system is given by the formula $\mathbf{u}(\mathbf{x})=\mathbf{v}(\mathbf{x})+\nabla h(\mathbf{x})$, where $\mathbf{v}(\mathbf{x})$ is a certain particular solution and $h(\mathbf{x})$ is an arbitrary harmonic function. Let us find the particular solution.

A differential form of degree 2: $\Omega\langle\mathbf{a},\mathbf{b}\rangle=\mathbf{b}\cdot\omega\langle\mathbf{a}\rangle$, corresponds to the antisymmetric tensor ω, while a differential form of degree 1: $U\langle\mathbf{a}\rangle=a\cdot\mathbf{u}$ corresponds to the vector $\mathbf{u}$ ($\mathbf{a},\mathbf{b}$ are arbitrary vectors in R^n).

The second equation of system (1.3) is equivalent to the equation $\partial U=\Omega$, where d is the external differentiation operator. However, the second equation of system (1.4) is the condition for the form Ω to be closed. By virtue of the Poincaré theorem /4/, it follows from the second equation of the automorphic system (1.3) that ($\varphi(\mathbf{x})$ is a certain function)

$$\mathbf{u}(\mathbf{x})=\int_0^1 t\omega(t\mathbf{x})\langle\mathbf{x}\rangle\,dt+\nabla\varphi(\mathbf{x}) \tag{2.1}$$

After substituting (2.1) into the first equation of system (1.3) and a number of calculations using the first equation of system (1.4), we obtain Poisson's equation to seek the function $\varphi(\mathbf{x})$

$$\Delta\varphi(\mathbf{x})=-\frac{\lambda+\mu}{\lambda+2\mu}\theta(\mathbf{x})+2\int_0^1 t\theta(t\mathbf{x})\,dt \tag{2.2}$$

Since $\theta(\mathbf{x})$ is a harmonic function, the right-hand side of this equation is also a harmonic function.

Lemma. If $\psi(\mathbf{x})$ is a harmonic function then the equation

$$\Delta\varphi(\mathbf{x})=\psi(\mathbf{x}) \tag{2.3}$$

has a particular solution of the form

$$\varphi(\mathbf{x})=\frac{1}{4}|\mathbf{x}|^2\int_0^1 t^{n/2-1}\psi(t\mathbf{x})\,dt \tag{2.4}$$

Proof. We will seek the solution of (2.3) in the form

$$\varphi(\mathbf{x})={}^1/_4\,|\mathbf{x}|^2 f(\mathbf{x})$$

where $f(\mathbf{x})$ is a harmonic function. Substitution into (2.3) results in a first-order equation. Introduction of the auxiliary function $g(\tau)=f(\tau\mathbf{x})$ yields an ordinary differential equation whose solution, bounded at the point $\tau=0$ has the form

$$g(\tau)=\tau^{-n/2}\int_0^\tau t^{n/2-1}\psi(t\mathbf{x})\,dt$$

We find the function f from the formula $f(x) = g(1)$; it is obviously harmonic.

We will now obtain the particular solution of (2.2) according to the lemma by substituting the right-hand side of this equation into (2.4). The repeated integral occurring here is transformed into a one-dimensional integral. After this, the general solution of the automorphic system is given by the formulas

$$\mathbf{u}(\mathbf{x}) = \int_0^1 t\omega(t\mathbf{x})\langle\mathbf{x}\rangle\,dt - \frac{1}{n-4}\nabla\Big\{|\mathbf{x}|^2 \times \tag{2.5}$$

$$\Big[\frac{(\lambda+\mu)n+4\mu}{4(\lambda+2\mu)}\int_0^1 t^{n/2-1}\theta(t\mathbf{x})\,dt - \int_0^1 t\theta(t\mathbf{x})\,dt\Big]\Big\} + \nabla h(\mathbf{x}), \quad n \neq 4$$

$$\mathbf{u}(\mathbf{x}) = \int_0^1 t\omega(t\mathbf{x})\langle\mathbf{x}\rangle\,dt - \frac{1}{4}\nabla\Big\{|\mathbf{x}|^2\Big[\frac{\lambda+\mu}{\lambda+2\mu}\int_0^1 t\theta(t\mathbf{x})\,dt + \tag{2.6}$$

$$2\int_0^1 t\theta(t\mathbf{x})\ln t\,dt\Big]\Big\} + \nabla h(\mathbf{x}), \quad n = 4$$

where $h(\mathbf{x})$ is an arbitrary harmonic function.

For $n = 2$ formula (2.5) is the known Kolosov-Muskhelishvili formula. For $n > 2$ formulas (2.5) and (2.6) can be considered as multidimensional analogues of this formula. They enable solutions of the Lamé equations to be obtained from the solutions θ, ω of the resolving system (1.4).

3. Group property of the resolving system. We will seek the operator allowed by the resolving system (1.4) in the form

$$\xi\cdot d_\mathbf{x} + \eta_0\partial_0 + \eta\partial_\omega = \xi_m\frac{\partial}{\partial x_m} + \eta_0\frac{\partial}{\partial\theta} + \eta_{mp}\frac{\partial}{\partial\omega_{mp}} \tag{3.1}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, η_0, $\eta = (\eta_{mp})$ are the required functions of the variables $\mathbf{x}$, θ, $\omega = (\omega_{mp})$.

In this case the solution of the sytem of governing equations depends substantially on the dimensionality n.

For $n = 2$ system (1.4) agrees with the Cauchy-Riemann system and allows of an infinite conformal group. The coordinates ξ_1, ξ_2, η_0, η_{12} $(\eta_{21} = -\eta_{12})$ of operators of the form (3.1) for this group are defined as follows: the pairs (ξ_1, ξ_2) and (η_0, η_{12}) are solutions of the Cauchy-Riemann system in both the independent variables x_1, x_2 and the functions θ, ω_{12}.

For $n > 3$ the principal group of system (1.4) is a similarity group of Euclidean n-space. The coordinates of the operators (3.1) forming its Lie algebra are

$$\xi = A\langle\mathbf{x}\rangle + \alpha\mathbf{x} + \mathbf{a}, \; \eta_0 = \beta\theta, \; \eta = \beta\omega + A\omega - \omega A$$

Here A is an arbitrary antisymmetric transformation of the space R^n, $\mathbf{a}$ is an arbitrary vector from R^n, and α, β are arbitrary real numbers. These operators are a continuation of the principal group operators of the Lamé Eqs.(1.1) in the tensor ω.

For $n = 3$ it is convenient to pass from the antisymmetric tensor ω to its corresponding vector $\boldsymbol{\omega} = \operatorname{rot}\mathbf{u}$. The resolving system (1.4) is written in the form

$$\nabla\theta - \operatorname{rot}\boldsymbol{\omega} = 0, \; \operatorname{div}\boldsymbol{\omega} = 0 \tag{3.2}$$

The Lie algebra L_{14} of the principal group operators of this system has the basis

$$\mathbf{X}_1 = \partial_\mathbf{x}, \quad \mathbf{X}_2 = \mathbf{x}\times\partial_\mathbf{x} + \boldsymbol{\omega}\times\partial_\omega$$
$$\mathbf{X}_3 = -|\mathbf{x}|^2\partial_\mathbf{x} + 2\mathbf{x}(\mathbf{x}\cdot\partial_\mathbf{x}) + (\mathbf{x} + \boldsymbol{\omega} - 2\theta\mathbf{x})\partial_0 -$$
$$2\mathbf{x}(\boldsymbol{\omega}\cdot\partial_\omega) + \mathbf{x}\times(\boldsymbol{\omega}\times\partial_\omega - \theta\partial_\omega), \quad X_4 = \mathbf{x}\cdot\partial_\mathbf{x}$$
$$\mathbf{X}_5 = \boldsymbol{\omega}\partial_\theta - \theta\partial_\omega - \boldsymbol{\omega}\times\partial_\omega, \quad X_6 = \theta\partial_\theta + \boldsymbol{\omega}\cdot\partial_\omega$$

(there are three scalars for each of the vector operators).

The subalgebra of L_{14} with basis $\mathbf{X}_1, \mathbf{X}_2 + {}^1/_2\mathbf{X}_5$, $\mathbf{X}_3$, $X_4 - X_6$ and the subalgebra of L_4 with the basis $\mathbf{X}_5, X_6$ are ideals of the algebra L_{14} which can be represented in the form of a direct sum: $L_{14} = L_{10} \oplus L_4$. The Lie algebra L_{10} is isomorphic to the Lie algebra of the conformal group of three-dimensional Euclidean space. Finite transformations conserving system (3.2) correspond to operators of the algebra: $\mathbf{X}_1$ is the translation in $\mathbf{x}$, $\mathbf{X}_2$ is the joint rotation in the spaces $R^3(\mathbf{x})$ and $R^3(\boldsymbol{\omega})$, X_4 is the uniform extension in $\mathbf{x}$, $\mathbf{X}_5$ is the rotation in the space $R^4(\theta, \boldsymbol{\omega})$, and X_6 is the uniform extension in θ, $\boldsymbol{\omega}$.

Let us consider the transformations generated by the generalized inversion operator $\mathbf{X}_3 = (X_{31}, X_{32}, X_{33})$. We write the for the operator X_{31} say (α is a real parameter):

$$x_1' = (x_1 - \alpha\,|\,\mathbf{x}\,|^2)/\zeta,\ x_m' = x_m/\zeta\ (m = 2, 3)$$
$$\mathrm{U}' = \sqrt{\zeta}B\mathrm{U};\ \zeta = (1 - \alpha x_1)^2 + \alpha^2(x_2^2 + x_3^2)$$
$$\mathrm{U} = \begin{Vmatrix} \theta \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{Vmatrix},\quad B = \begin{Vmatrix} 1-\alpha x_1 & 0 & -\alpha x_3 & \alpha x_2 \\ 0 & 1-\alpha x_1 & -\alpha x_2 & -\alpha x_3 \\ \alpha x_3 & \alpha x_2 & 1-\alpha x_1 & 0 \\ -\alpha x_2 & \alpha x_3 & 0 & 1-\alpha x_1 \end{Vmatrix}$$

Therefore, the group reduction executed for the Lamé equations enables the symmetries which the Lamé equations themselves do not possess in the plane and three-dimensional cases that have physical meaning to be revealed. The principal group of the resolving system (3.2) is broader than the principal group of the Lamé equations. The new operators X_{31}, X_{32}, X_{33} generating the conformal transformations expand the possibilities of constructing classes of particular solutions by using the group being allowed for the system (3.2), and therefore, for the Lamé equations also, by virtue of (2.5).

Leaving aside the plane cases which have been studied in detail in the theory of elasticity, we will henceforth examine the resolving system only for $n = 3$.

4. Kelvin transformations. The presence of the generalized inversion operators X_3 indicates the existence of transformations of the Kelvin transformation type for the resolving system (3.2). By analogy with the latter we give the following definition:

Definition. We call a Kelvin transformation for system (3.2) the transformation K having the following properties:

1) It sets the function $\mathrm{U}: R^3 \to R^4$ in correspondence with the function defined by the equality $\mathrm{U}'(\mathbf{x}) = K(\mathbf{x}) \times \mathrm{U}(\mathbf{x}/|\,\mathbf{x}\,|^2)$, where $K(\mathbf{x})$ is a 4x4 matrix;

2) If $\mathrm{U}(\mathbf{x})$ is a solution of system (3.2), then $\mathrm{U}'(\mathbf{x})$ is also a solution of this system.

Theorem. All Kelvin transformations for system (3.2) are described by the following expression:

$$K(\mathbf{x}) = \alpha_1 K_1(\mathbf{x}) + \alpha_2 K_2(\mathbf{x}) + \alpha_3 K_3(\mathbf{x}) + \alpha_4 K_4(\mathbf{x})$$
$$K_1(\mathbf{x}) = \frac{1}{|\mathbf{x}|^3}\begin{Vmatrix} x_1 & 0 & x_3 & -x_2 \\ 0 & x_1 & x_2 & x_3 \\ x_3 & x_2 & -x_1 & 0 \\ -x_2 & x_3 & 0 & -x_1 \end{Vmatrix},\quad K_2(\mathbf{x}) = \frac{1}{|\mathbf{x}|^3}\begin{Vmatrix} x_2 & -x_3 & 0 & x_1 \\ -x_3 & -x_2 & x_1 & 0 \\ 0 & x_1 & x_2 & x_3 \\ x_1 & 0 & x_3 & -x_2 \end{Vmatrix}$$
$$K_3(\mathbf{x}) = \frac{1}{|\mathbf{x}|^3}\begin{Vmatrix} x_3 & x_2 & -x_1 & 0 \\ x_2 & -x_3 & 0 & x_1 \\ -x_1 & 0 & -x_3 & x_2 \\ 0 & x_1 & x_2 & x_3 \end{Vmatrix},\quad K_4(\mathbf{x}) = \frac{1}{|\mathbf{x}|^3}\begin{Vmatrix} 0 & -x_1 & -x_2 & -x_3 \\ x_1 & 0 & x_3 & -x_2 \\ x_2 & -x_3 & 0 & x_1 \\ x_3 & x_2 & -x_1 & 0 \end{Vmatrix}$$

(α_m $(m = 1, 2, 3, 4)$ are arbitrary real numbers).

The proof consists of a direct evaluation of the matrix $K(\mathbf{x})$ on the basis of the definition given above.

We note certain properties of the basis Kelvin transformations K_m. To write their composition we introduce matrices corresponding to the rotation operators

$$Q_1 = \begin{Vmatrix} P_1 & O \\ O & P_1 \end{Vmatrix},\quad Q_2 = \begin{Vmatrix} O & P_2 \\ -P_2 & O \end{Vmatrix},\quad Q_3 = \begin{Vmatrix} O & P_3 \\ -P_3 & O \end{Vmatrix}$$
$$P_1 = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix},\quad P_2 = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix},\quad P_3 = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}$$

The compositions of the basis Kelvin transformations are determined by the matrices (I is the identity transformation)

$$M = \begin{Vmatrix} I & Q_3 & -Q_2 & -Q_1 \\ -Q_3 & I & Q_1 & -Q_2 \\ Q_2 & -Q_1 & I & -Q_3 \\ -Q_1 & -Q_2 & -Q_3 & -I \end{Vmatrix}$$

Its elements are the transformations $M_{pq} = K_p K_q$.

It follows from the form of the matrix M, in particular, that any three basis Kelvin transformations can be obtained from the fourth because of the composition of this latter with the transformations Q_1, Q_2, Q_3, for example $K_p = Q_p K_4$ $(p = 1, 2, 3)$.

5. **Complex variables.** The resolving system (3.2) consists of four scalar equations for four unknown scalar functions: the function θ characterizing the volumetric deformation and three components of the vector $\omega = (\omega_1, \omega_2, \omega_3)$. Introduction of the complex dependent and independent variables

$$u = \theta + i\omega_3, \quad v = \omega_2 - i\omega_1 \tag{5.1}$$
$$x = {}^1/_2\,(x_1 + ix_2), \quad y = {}^1/_2\,(-x_1 + ix_2), \quad t = x_3$$

enables us to write system (3.2) in the more compact form

$$\partial_t u = \partial_x v, \quad \partial_t v = \partial_y u \tag{5.2}$$
$$\partial_x = \partial_{x_1} - i\partial_{x_2}, \quad \partial_y = -\partial_{x_1} - i\partial_{x_2}$$

where ∂_x, ∂_y are formal differentiation operators.

Let us investigate the properties of system (5.2) by considering x, y, t as independent complex variables.

If u, v is a solution of system (5.2) with the independent complex variables x, y, t, then a solution of system (3.2) is obtained therefrom by means of (5.1). Conversely, if in any solution of the system (3.2) we change we complex functions u, v by means of (5.1), continue them analytically in C^3 and make a linear substitution in the complex independent variables by means of (5.1), then a solution of system (5.2) is obtained.

System (5.2) admits of an 11-parameter fundamental Lie transformation group of the space $C^5\,(x, y, t, u, v)$ generated by the operators

$$Y_1 = \partial_x, \; Y_2 = \partial_y, \; Y_3 = \partial_t$$
$$Y_4 = 2x\partial_x + t\partial_t + v\partial_v, \; Y_5 = 2y\partial_y + t\partial_t + u\partial_u$$
$$Y_6 = t\partial_x + 2y\partial_t - u\partial_v, \; Y_7 = t\partial_y + 2x\partial_t - v\partial_u$$
$$Y_8 = t^2\partial_x + 4y^2\partial_y + 4ty\partial_t - 2yu\partial_u - 2\,(tu + 3yv)\partial_v$$
$$Y_9 = 4x^2\partial_x + t^2\partial_y + 4tx\partial_t - 2(tv + 3xu)\partial_u - 2xv\partial_v$$
$$Y_{10} = tx\partial_x + ty\partial_y + {}^1/_2\,(t^2 + 4xy)\partial_t - (tu + yv)\partial_u -$$
$$(tv + xu)\partial_v, \quad Y_{11} = u\partial_u + v\partial_v$$

It is convenient to use the complex system (5.2) to obtain exact solutions of system (3.2).

We will present some examples of exact solutions of system (5.2) and we will indicate the nature of the corresponding solutions of system (3.2).

1°. A solution invariant under a subgroup with operator Y_{10} is a solution of rank 2 and has the form

$$u = -2y^{-1/2}f\,(ry) + tx^{-3/2}g\,(rx)$$
$$v = ty^{-3/2}f\,(ry) - 2x^{-1/2}g\,(rx)$$
$$r = (t^2 - 4xy)^{-1}$$

where f, g are arbitrary analytic functions. The corresponding solution of system (3.2) will be invariant under the subgroup generated by the operator X_{33}.

2°. A solution invariant under the subgroup defined by the operator Y_8 is also of rank 2. It is

$$u = 4y^{-1/2}\,[ty^{-1}f'\,(z) + g\,(z)]$$
$$v = -y^{-3/2}\,\{f\,(z) + 2t\,[ty^{-1}f'\,(z) + g\,(z)]\}$$
$$z = y^{-1}\,(t^2 - 4xy)^{-1}$$

where f, g are arbitrary analytic functions. The solution of system (3.2) therefore obtained by means of (5.1) is not invariant under any of the subgroups allowed by this system.

3°. We will consider simple waves of system (5.2). Their parametric representation with the complex parameter $\varepsilon = \varepsilon\,(x, y, t)$ has the form

$$u = u\,(\varepsilon), \quad v = v\,(\varepsilon) \tag{5.3}$$

Substitution into system (5.2) yields a system of equations connecting the desired functions (5.3) and the desired parameter

$$\varepsilon_t u' = \varepsilon_x v', \quad \varepsilon_t v' = \varepsilon_y u'$$

where the prime denotes the derivative with respect to ε. Analysis of this system shows that $u\,(\varepsilon)$ and $v\,(\varepsilon)$ remain arbitrary analytic functions while the wave parameter is defined implicitly by the equation

$$xu'^2 + yv'^2 + tu'v' = f(\varepsilon) \tag{5.4}$$

where f is an arbitrary analytic function.

To obtain the corresponding solution of system (3.2) it is convenient to select $\varepsilon = v$ as the parameter. Changing to real variables by means of (5.1) we obtain the double wave of system (3.2)

$$\theta = \varphi(\omega_1, \omega_2), \quad \omega_3 = \psi(\omega_1, \omega_2)$$

in which the functions φ, ψ satisfy the Cauchy-Riemann conditions.

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THE GEOMETRICAL CHARACTERISTICS OF EQUALLY-STRONG BOUNDARIES OF ELASTIC BODIES*

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The necessary conditions for the existence of systems of surfaces or plane curves of special shape determined from mechanical considerations, by potential theory methods, are found, a number of integral identities is constructed, and certain modifications of the Robin problem are solved.

1. A linearly elastic homogeneous and istoropic three-dimensional domain S of the space E is considered which is weakened by a set of m non-intersecting closed cavities S_k^- with smooth boundaries Γ_k $(k = 1, 2, \ldots, m)$ and is loaded by remote forces P_i $(i = 1, 2, 3)$ along the axes of an $X_1X_2X_3$ Cartesian coordinate system, G, ν are the elastic moduli of the medium, and $I_1(x)$, $I_2(x)$ are stress tensor invariants at an arbitrary point $x = (x_1, x_2, x_3)$.

The boundary $\Gamma = \bigcup \Gamma_k$ is called equally-strong for a given load [I] if the identity $I_1(\xi) = \text{const}$ holds at any of its points $\xi = (\xi_1, \xi_2, \xi_3)$. The constant on the right-hand side equals $P_1 + P_2 + P_3 = P$. It is proved in [I] that such a boundary minimizes the maximum value, over the domain, of the local Mises plasticity criterion $F(x) = I_1^2(x) - 3I_2(x)$, thereby being the solution of the following optimal control type problem:

$$\max_{x \in (S+\Gamma)} F(x) \to \min_{\{\Gamma\}} \tag{1.1}$$

Since the function $F(x)$ is invariant under a similarity transformation of the coordinates, the optimal boundary according to (1.1), if it exists, is not defined uniquely, but to at least the accuracy of a scale given by an arbitrary factor C. Indeed, the class of solutions is significantly broader in many cases, which is utilized substantially in Sect.3.

It has been established /1/ that the components of the displacement vector $\mathbf{u}(x)$ of the state of stress corresponding to a perturbation induced in the homogeneous field of cavities are harmonic functions in the domain S that decrease at infinity as $O(|x|^{-2})$, take values on the optimal boundary that are proportional to the corresponding coordinate at the point

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